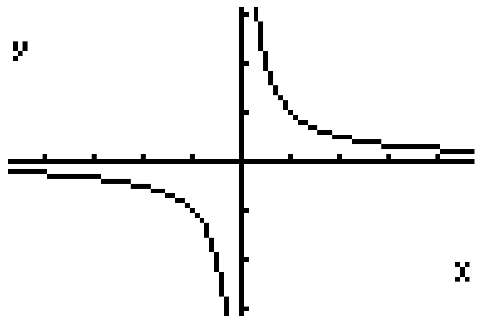
Infinite Limits and Asymptotes

-We now need to look at limits as they travel without bound.

-The first classic example is $f(x) = \frac{1}{x}$



-We want to consider $\lim_{x \to 0} \frac{1}{x}$

-We can't simply substitute to find the limit.
-Since limits are analytic we need to look at what happens as x→0

$$\frac{1}{1} = 1 \qquad \frac{1}{.5} = 2 \qquad \frac{1}{.25} = 4 \qquad \frac{1}{.125} = 8$$

-As the denominator gets closer to zero we see that the value gets larger.
-If we would use a negative value we would see the values get increasingly negative.

-Because of this we know that the two-sided limit of $f(x) = \frac{1}{x}$ is undefined.

-But $\lim_{x \to 0^-} \frac{1}{x} = -\infty$ and $\lim_{x \to 0^+} \frac{1}{x} = \infty$

-This means that the function either increases or decreases without bound as $x \to 0$ from either side.

-Looking at the graph and knowing the limits go to positive and negative infinity tell us there is a vertical asymptote.

Checking for Vertical Asymptotes

If $\lim\limits_{x\to c} f\left(x\right) = \pm\infty$ then $x = c$ is a vertical asymptote.

-The other type of limit involving infinity is when the limits are evaluated as $x \to \pm\infty$.

-We can again use the model $f\left(x\right) = \dfrac{1}{x}$. In order to now consider the limits $\lim\limits_{x\to\pm\infty} \dfrac{1}{x}$ we can't simply substitute in infinity.

-To solve we need to start thinking analytically.

$$\frac{1}{10}, \frac{1}{100}, \frac{1}{1{,}000}, \ldots, \frac{1}{1{,}000{,}000}$$

-If we consider that the values in the denominator will continue to get larger and larger what value does this approach?

-This works for the function $f\left(x\right) = \dfrac{1}{x}$ but what about other functions?

-Consider the polynomial function

$$f\left(x\right) = x^3 + 2x + 3$$

-Does adding 3 really matter when $x = 1000$?

-If $x = 1000$ then:

$$x^3 = 1{,}000{,}000{,}000$$
$$2x = 2000$$

-So does adding $2x$ matter?

NO

-This means that $g(x) = x^3$ is an <u>end behavior model</u> for $f(x)$.

End Behavior Models

-Model the behavior of a function as x approaches infinity or negative infinity.

-For large values of x, $g(x)$ will model the behavior of $f(x)$ for large values of x.

-A function g is:

a right end behavior model for f iff $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$

a left end behavior model for f iff $\lim_{x\to-\infty} \frac{f(x)}{g(x)} = 1$

Check for End Behavior Model

-To check to see if a function is an end behavior model use the definition!

$$f(x) = 3x^3 - 5x + 9$$

-Pick $g(x) = 3x^3$ 'The most powerful term.

-Then test the left and the right

Right

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\frac{3x^3-5x+9}{3x^3}$$

$$=\lim_{x\to\infty}\frac{3x^3}{3x^3}-\frac{5x}{3x^3}+\frac{9}{3x^3}$$

$$=\lim_{x\to\infty}1-\frac{5}{3x^2}+\frac{3}{x^3}$$

$=1$ 'Passes

Right

$$\lim_{x\to-\infty}\frac{f(x)}{g(x)}=\frac{3x^3-5x+9}{3x^3}$$

-FOLLOWS SUIT

$=1$ 'Passes

-Since the left and right both equal 1 $g(x)$ is an end behavior model.

Using End Behavior Models to Evaluate Limits to Infinity

Evaluate $\lim_{x\to\infty}\frac{3x^3-5x+9}{5x^3+2x^2-7}$

-Choose a term that is the most powerful in both the numerator and denominator.

$$= \lim_{x\to\infty} \frac{3x^3 \cancel{-5x} \cancel{+9}}{5x^3 \cancel{+2x^2} \cancel{-7}}$$

$$= \lim_{x\to\infty} \frac{3\cancel{x^3}}{5\cancel{x^3}}$$

$$= \frac{3}{5}$$

Using EBM's

Evaluate $\lim_{x\to\infty} \frac{2x^5 + x^4 - x^2 + 1}{3x^2 - 5x + 7}$

$$= \lim_{x\to\infty} \frac{2x^5}{3x^2}$$

$$= \lim_{x\to\infty} \frac{2}{3}x^3 = \infty$$

Evaluate $\lim_{x\to\infty} \frac{2x^3 - x^2 + x - 1}{5x^3 + x^2 + x - 5}$

$$= \lim_{x\to\infty} \frac{2x^3}{5x^3}$$

$$= \frac{2}{5}$$

Checking for Horizontal Asymptotes

-If $\lim_{x\to+\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$ then the line $y = L$ is a horizontal asymptote of the graph of f.

Example

Find any horizontal asymptotes of $f(x) = \dfrac{9x^5 + 50x^2 + 800}{x^5 - 1000}$

$$\lim_{x\to+\infty} = \frac{9x^5 \cancel{+50x^2} \cancel{+800}}{x^5 \cancel{-1000}}$$

$$\lim_{x\to+\infty} = \frac{9\cancel{x^5}}{\cancel{x^5}} = 9$$

-Therefore there is a horizontal asymptote at $y = 9$. You can graph to confirm.

Example of a Vertical Asymptote

Find any vertical asymptotes of $f(x) = \dfrac{x^2 - 3x + 4}{x + 2}$

$$\lim_{x\to-2^-} = \frac{x^2 - 3x + 4}{x + 2}$$

$$= \frac{(-2.1)^2 - 3(-2.1) + 4}{(-2.1) + 2} = -147.1$$

$$= \frac{(-2.01)^2 - 3(-2.01) + 4}{(-2.01) + 2} = -1407.01$$

$$\lim_{x \to -2^-} = \frac{x^2 - 3x + 4}{x + 2} = -\infty$$

-Therefore $x = -2$ is a vertical asymptote.

Example

Evaluate $\lim_{x \to \infty} \frac{\sin x}{x}$

$$= \lim_{x \to \infty} \frac{\text{between } -1 \text{ and } 1}{\text{really really big}} = 0$$

Example

Find any asymptotes and describe the behavior of $f(x) = \frac{x^2 - 1}{2x + 4}$:

$$= \lim_{x \to -2} \frac{x^2 - 1}{2x + 4}$$

$$= \lim_{x \to -2^-} \frac{x^2 - 1}{2x + 4} = \frac{(-2.001)^2 - 1}{2(-2.001) + 4} = \frac{+}{-} = -\infty$$

$$= \lim_{x \to -2+} \frac{x^2 - 1}{2x + 4} = \frac{(-1.999)^2 - 1}{2(-1.999) + 4} = \frac{+}{+} = +\infty$$